*Journal of Statistical Physics, Vol. 88, Nos. 1/2, 1997*

# Half-Space Problem of the Boltzmann Equation for Charged Particles

**Ch. Dalitz**[1]

*Received August 14, 1996*

For two particular collision kernels, we explicitly solve the one-dimensional stationary half-space boundary value problem of the linear Boltzmann equation including a constant external field via an extension of Case's eigenfunction technique. In the first collision model we reproduce a solution recently obtained by Cercignani; in the second model the solution of the stationary boundary value problem is presented for the first time.

**KEY WORDS:** Boltzmann equation; boundary-value problem; half-range completeness.

## 1. INTRODUCTION

An ensemble of particles with mass $m$ and charge $e$, which move in a neutral host medium under the action of an external field $E$, can be described by a probability density $f(t, x, p)$ for finding a charged test particle at point $x$ and time $t$ with momentum $p$. This probability density satisfies the Boltzmann equation, which reads in one dimension

$$\left(\frac{\partial}{\partial t}+\frac{p}{m}\frac{\partial}{\partial x}+eE\frac{\partial}{\partial p}\right)f(t, x, p)=\mathcal{Q}f(t, x, p)$$

The collision operator $\mathcal{Q}$ is a linear integral operator, if interactions among the charge carriers themselves (via Pauli's principle or collisions) can be neglected.

In this article, we examine the stationary half-space problem for the Boltzmann equation, i.e., we assume the medium to be semiinfinite

[1] Fakultät für Physik, Universität Bielefeld, 33615 Bielefeld, Germany.



0022-4715/97/0700-0129$12.50/0 

($x \in (0, \infty)$). At the boundary $x = 0$ charged particles may be shot in with a given momentum distribution $\varphi$. Thus we seek the solution of

$$\left(\frac{p}{m}\frac{\partial}{\partial x} + eE\frac{\partial}{\partial p}\right) f(x, p) = \mathcal{Q} f(x, p) \tag{1}$$

with the boundary conditions

$$f(0, p) = \varphi(p) \quad \text{for} \quad p > 0 \quad \text{and} \quad \lim_{x \to \infty} f(x, p) = 0 \tag{2}$$

For zero electric field $E \equiv 0$ this problem has been studied extensively in the context of neutron diffusion in solids.[1] The most popular solution scheme, originally due to Case,[2] is an expansion of the solution in terms of "singular eigenfunctions." Completeness theorems for these eigenfunctions have been proven for the general equation $h(p)\,\partial_x f + \mathcal{A} f = 0$, under the assumption that $\mathcal{A}$ be a selfadjoint positive operator in some appropriate function space.[3] The positivity assumption may even be dropped.[4]

In the case of a nonzero electric field, these general theorems do not apply because the operator $\mathcal{A} = eE\partial_p - \mathcal{Q}$ is not selfadjoint. Nevertheless, an eigenfunction expansion might be possible for specific collision operators. In ref. 5 the author made an attempt in this direction and obtained a kind of half-range completeness for the RTA- (or BGK-) model in the limit of zero ambient temperature. This is not a very amazing result, since in this limit boundary-value problems generally can be solved analytically.[6]

In this article, we present two nontrivial collision models for which the half-space problem (1) (2) can be solved via an eigenfunction expansion. Under the assumption that the boundary value $\varphi$ be a Laplace transform, we prove half-range completeness of the eigenfunctions. Moreover, by straightforward insertion of the expansion coefficients into the eigenfunction representation of the solution, we obtain the solutions explicitly in terms of the boundary value $\varphi$, so that the restrictive assumptions upon $\varphi$ for the half-range completeness may be dropped.

## 2. THE EIGENFUNCTION EXPANSION

Inserting the separation ansatz $f_\lambda(x, p) := e^{-\lambda x} g_\lambda(p)$ into Eq. (1), we obtain for the functions $g_\lambda$ the "eigenvalue" equation

$$(\mathcal{Q} - eE\partial_p)\, g_\lambda(p) = -\lambda \frac{p}{m}\, g_\lambda(p) \tag{3}$$

Since Eq. (1) is linear, any superposition of solutions $f_\lambda(x, p)$ is a solution of (1) too. Thus we assume the solution of (1) to be of the form

$$f(x, p) = \int d\lambda\, A(\lambda)\, f_\lambda(x, p) = \int d\lambda\, A(\lambda)\, e^{-\lambda x}\, g_\lambda(p) \tag{4}$$

The integrals run over all $\lambda$ for which a $L_1$-integrable solution of (3) exists. Because of the boundary condition at infinity, only $\lambda$ with a positive real part are admitted. In consequence, the force $eE$ is assumed to point in the negative $x$-direction, that is $eE < 0$.[5] The expansion coefficients $A(\lambda)$ must be determined from the boundary value $\varphi(p)$. Setting $x = 0$ in (4) yields

$$f(0, p) = \varphi(p) = \int d\lambda\, A(\lambda)\, g_\lambda(p) \qquad \text{for} \quad p > 0 \tag{5}$$

Now the task is to determine the class of boundary values $\varphi$ for which this equation has a solution $A(\lambda)$. In order to make this problem tractable, we restrict ourselves to real values of $\lambda$ (which will turn out to be sufficient).

The eigenfunctions $g_\lambda$ depend strongly on the collision operator $\mathcal{Q}$, which has the general form

$$\mathcal{Q} f(x, p) = M(p) \int dp'\, K(p, p')\, f(x, p') - f(x, p) \int dp'\, K(p', p)\, M(p') \tag{6}$$

where the kernel $K(p, p')$ is symmetric and positive and $M(p) = (2\pi\theta m)^{-1/2} e^{-p^2/2\theta m}$ is the Maxwellian at the temperature $\theta$ of the neutral medium, which is assumed to be in equilibrium. We are going to examine the following specific collision kernels:

(a) $K(p, p') = \rho\, |p| \cdot |p'|$. In this model, sometimes called "constant free path model," the first integral on the right hand side of Eq. (6) is closely related to the current, which was the starting point for Cercignani's solution of the stationary boundary value problem.[7] His solution scheme even may be extended to $x$-dependent electric fields. Because of these amazing results, we will call this model "Cercignani's model."

(b) $K(p, p') = \rho\, |p - p'|$. Since this model has been studied by Piasecki in the presence of an electric field, we will associate this model with his name. In ref. 8 the inhomogeneous initial value problem has been solved in the zero temperature limit, in ref. 9 the homogeneous stationary solution has been obtained and in ref. 10 the homogeneous initial value problem has been solved. Up to now, no solution of the stationary boundary value problem is known.

It is interesting to note that in the zero temperature limits $\rho\,|p|\,M(p) \to \delta(p)$ for (a) and $\rho M(p) \to \delta(p)$ for (b) both models coincide. For this limiting model, the stationary boundary value problem has been explicitly solved in ref. 6, Section 5b).

## 3. APPLICATION TO CERCIGNANI'S MODEL

In this model we have $K(p, p') = \rho\,|p| \cdot |p'|$. Thus, after transition to the dimensionless quantities

$$\hat{p} := \frac{p}{\sqrt{\theta m}}, \qquad \hat{x} := x\rho m \sqrt{\frac{2\theta m}{\pi}}, \qquad \varepsilon := -\frac{eE}{\rho \theta m}\sqrt{\frac{\pi}{2\theta m}} > 0$$

the eigenvalue equation (3) reads (for convenience the hats of the new variables are dropped)

$$(\varepsilon \partial_p - |p| + \lambda p)\, g_\lambda(p) = -|p| \sqrt{\frac{\pi}{2}}\, M(p) \int dp'\, |p'|\, g_\lambda(p') \tag{7}$$

where $M(p) = (2\pi)^{-1/2} e^{-p^2/2}$ is the (dimensionless) Maxwellian.

### 3.1. Determination of the λ-Spectrum

In order to examine for which $\lambda$ an integrable solution of (7) exists, let us split $g_\lambda(p) =: \Theta(p)\, g_\lambda^+ + \Theta(-p)\, g_\lambda^-$, where $\Theta$ denotes Heavisides's step-function. To get rid of the integral on the right hand side of (7), we normalize $\int |p|\, g_\lambda = 1$. Then (7) reads for $\pm p > 0$

$$(\varepsilon \partial_p + p(\lambda \mp 1))\, g_\lambda^\pm = \mp \frac{p}{2} e^{-p^2/2} \tag{8}$$

For $\lambda > 1$, the general solution of (8) is

$$g_\lambda^\pm(p) = e^{-(\lambda \mp 1) p^2/2\varepsilon} \left( C_\lambda^\pm \mp \frac{1}{2\varepsilon} \int_0^p dq\, q e^{(\lambda \mp 1 - \varepsilon) q^2/2\varepsilon} \right) \tag{9}$$

The continuity condition $g_\lambda^+(0) = g_\lambda^-(0)$ leads to $C_\lambda^+ = C_\lambda^-$, and from the normalization $\int |p|\, g_\lambda = 1$ we find $C_\lambda^+ = \lambda/3\varepsilon$. Evaluating the integral on the right hand side of (9), we obtain for $\lambda > 1$:

$$g_\lambda^\pm(p) = \frac{\lambda}{2\varepsilon} e^{-(\lambda \mp 1) p^2/2\varepsilon} \pm \frac{e^{-(\lambda \mp 1) p^2/2\varepsilon} - e^{-p^2/2}}{2(\lambda \mp 1 - \varepsilon)} \tag{10}$$

We remark that in the limit $\lambda \to \varepsilon \pm 1$ (10) remains well defined, as can be seen by application of de l'Hospital's rule.

For $\lambda < 1$, the solution for negative momenta, $g_\lambda^-$, is again given by (9). However, for positive $p$ (9) grows exponentially as $p \to \infty$ if $C_\lambda^+$ is not properly chosen. Hence, the only acceptable solution is

$$g_\lambda^+(p) = e^{-(\lambda-1)p^2/2\varepsilon} \frac{1}{2\varepsilon} \int_p^\infty dq\, q e^{(\lambda-1-\varepsilon)q^2/2\varepsilon} = \frac{e^{-p^2/2}}{2(\varepsilon - \lambda + 1)}$$

From the continuity condition $g_\lambda^+(0) = g_\lambda^-(0)$ we find $C_\lambda^- = 1/2(\varepsilon - \lambda + 1)$, and the normalization $\int |p|\ g_\lambda = 1$ leads to

$$1 = \frac{1}{\lambda + 1} \cdot \frac{\varepsilon + 1}{\varepsilon - \lambda + 1}$$

This only holds if $\lambda = 0$ or $\lambda = \varepsilon$.

In summary, the $\lambda$-spectrum consists of the continuous interval $(1, \infty)$, with the corresponding eigenfunctions (10), and the two discrete points $\lambda = 0$, with the homogeneous solution as corresponding eigenfunction, and $\lambda = \varepsilon$, with the corresponding eigenfunction $g_\varepsilon(p) = \sqrt{\pi/2}\, M(p)$ which is proportional to the Maxwellian. If $\varepsilon > 1$, the latter eigenvalue lies in the continuous region.

## 3.2. Half-Range Completeness

Now let us consider Eq. (5) and examine which boundary values $\varphi$ can be represented as superpositions of eigenfunctions $g_\lambda$. Because of the boundary condition at infinity (2), no contribution of the homogeneous solution ($\lambda = 0$) may be present and hence the integral in (5) should be an integral over $(1, \infty)$ plus a single contribution of the discrete eigenfunction $M(p)$. The following theorem, which is in analogy to the half-range completeness of Case's singular eigenfunctions[2] (though for a much smaller class of functions), gives the class of functions $\varphi$ for which such a representation holds. Surprisingly, there is a singular contribution of the Maxwellian even for $\varepsilon > 1$.

**Theorem.** Each function $\varphi\colon \mathbb{R}^+ \to \mathbb{R}$ with $\int_0^\infty dp\, p\varphi < \infty$, that is a Laplace transform of any tempered distribution $\Phi$

$$\varphi(p) = \int_0^\infty d\mu\, \Phi(\mu)\, e^{-\mu p^2/2\varepsilon} \tag{11}$$

is a superposition of eigenfunctions

$$\varphi(p) = a_\varepsilon g_\varepsilon(p) + \int_1^\infty d\lambda\, A(\lambda)\, g_\lambda(p) \tag{12}$$

with expansion coefficients

$$a_\varepsilon = 2\varepsilon \int_0^\infty \frac{d\mu}{\mu} \mathscr{P} \frac{\Phi(\mu)}{\mu - \varepsilon + 1}, \qquad A(\lambda) = \frac{2\varepsilon(\lambda - \varepsilon - 1)}{(\lambda - 1)} \mathscr{P} \frac{\Phi(\lambda - 1)}{\lambda - \varepsilon} \tag{13}$$

where the tempered distribution $T(\mu) = \mathscr{P}\Phi(\mu)/(\mu - \varepsilon + 1)$ is a particular solution of the (distribution-) equation $(\mu - \varepsilon + 1) \cdot T(\mu) = \Phi(\mu)$.

*Remarks.* (a) Accoding to Hörmander's theorem,[11] there is a solution $T$ of $(\mu - \varepsilon + 1) \cdot T = \Phi$ for any tempered distribution $\Phi$. If $\varepsilon < 1$, we simply have $T = \Phi/(\mu - \varepsilon + 1)$. If $\varepsilon > 1$ and $\Phi(\mu)$ is Hölder-continuous at $\mu = \varepsilon - 1$, we can interprete the symbol $\mathscr{P}$ as "principal value." For general $\Phi$, the solution $T$ might be more complicated.

(b) From the assumption $\infty > \int_0^\infty dp\, p\varphi = \varepsilon \int_0^\infty d\mu\, \Phi/\mu$, which simply demands that $\varphi$ be in the definition domain of the collision operator, it follows that $\Phi/\mu$ is integrable on $(0, \infty)$, so that (3) is well defined.

(c) Assumption (11) is a restriction upon the admitted boundary values, of course. Conditions under which (11) holds are given in ref. 12; the most serious restriction is that $\varphi$ be analytic in a complex halfplane.

*Proof.* Let us start with the assumption that $\varphi$ is a superposition of the continuous eigenfunctions alone and then prove that a contribution of the discrete eigenfunction $g_\varepsilon(p) = \sqrt{\pi/2}\, M(p)$ must be added. Thus, let us try to solve for $A(\lambda)$:

$$\begin{aligned} \varphi(p) &= \int_1^\infty d\lambda\, A(\lambda)\, g_\lambda(p) \\ &= \int_0^\infty d\lambda\, A(\lambda + 1)\, g_{\lambda+1}(p) \\ &= \int_0^\infty d\lambda\, A(\lambda + 1) \frac{\lambda(\lambda + 1 - \varepsilon)\, e^{-\lambda p^2/2\varepsilon} - \varepsilon e^{-p^2/2}}{2\varepsilon(\lambda - \varepsilon)} \end{aligned} \tag{14}$$

If we insert (11) into (14), multiply with $e^{p^2/2}$, differentiate with respect to $p$ and use uniqueness of the Laplace transform, we find

$$\Phi(\lambda) = A(\lambda + 1) \frac{\lambda(\lambda + 1 - \varepsilon)}{2\varepsilon(\lambda - \varepsilon)} \tag{15}$$

Now we must distinguish two cases, depending on the electric field strength:

$\varepsilon < 1$. In this case the numerator on the right hand side of Eq. (15) has no zero and the solution of (15) is

$$A(\lambda) = \Theta(\lambda - 1)\, 2\varepsilon \frac{(\lambda - \varepsilon - 1)\, \Phi(\lambda - 1)}{(\lambda - \varepsilon)(\lambda - 1)}$$

However, inserting this result into (14) we obtain

$$\varphi(p) = \varphi(p) - \varepsilon e^{-p^2/2} \int_0^\infty d\lambda \frac{\Phi(\lambda)}{\lambda(\lambda + 1 - \varepsilon)}$$

We see that in (14) the term $a_\varepsilon g_\varepsilon(p)$, where $a_\varepsilon$ is given by (13), must be added. This results in our theorem.

$\varepsilon > 1$. In this case the numerator on the right hand side of Eq. (15) has a zero at $\lambda = \varepsilon - 1$ and consequently, the solution of (15) is a distribution rather than a function. In the space of distributions, the general solution of (15) is

$$A(\lambda) = \Theta(\lambda - 1)\, 2\varepsilon \frac{\lambda - \varepsilon - 1}{(\lambda - 1)} \mathscr{P} \frac{\Phi(\lambda - 1)}{\lambda - \varepsilon} + c_\varepsilon \delta(\lambda - \varepsilon)$$

where $c_\varepsilon$ is an arbitrary constant and $\mathscr{P}\Phi(\lambda - 1)/(\lambda - \varepsilon)$ denotes a particular solution $T(\lambda)$ of the equation $\Phi(\lambda - 1) = T(\lambda) \cdot (\lambda - \varepsilon)$. Existence of such a solution is assured by Hörmander's theorem.[11] Insertion of this result into (14) yields

$$\varphi(p) = \varphi(p) - \varepsilon e^{-p^2/2} \int_0^\infty \frac{d\lambda}{\lambda} \mathscr{P} \frac{\Phi(\lambda)}{\lambda + 1 - \varepsilon} + c_\varepsilon \frac{e^{-p^2/2}}{2}$$

If we identify $c_\varepsilon$ with $a_\varepsilon$ in (13), this equality holds and we obtain our theorem. ∎

## 3.3. Explicit Form of the Solution

For any boundary value $\varphi$ satisfying the conditions of the half-range completeness theorem, the solution of the stationary half-space problem is (see (4))

$$f(x, p) = a_\varepsilon \tfrac{1}{2} e^{-p^2/2} e^{-\varepsilon x} + \int_1^\infty d\lambda\, A(\lambda)\, e^{-\lambda x} g_\lambda(p) \tag{16}$$

By straightforward insertion of the eigenfunctions (10) and expansion coefficients (13) into (16), we can express $f(x, p)$ explicitly in terms of $\varphi$. Remarkably, in the resulting expression the assumption upon $\varphi$ that it be a Laplace transform may be dropped and we obtain the solution for any $\varphi$ with $\int_0^\infty dp\, p\varphi < \infty$.

Let us split $f(x, p) = \Theta(p)\, f^+ + \Theta(-p)\, f^-$. Insertion into (16) yields for positive momenta

$$f^+(x, p) = e^{-x} \int_0^\infty d\lambda\, \Phi(\lambda)\, e^{-\lambda(x + p^2/2\varepsilon)} + e^{-\varepsilon x - p^2/2}\varepsilon \int_0^\infty \frac{d\lambda}{\lambda}\, \mathscr{P} \frac{\Phi(\lambda)}{\lambda + 1 - \varepsilon} (1 - e^{-x(\lambda + 1 - \varepsilon)})$$

If we write

$$\frac{\varepsilon}{\lambda(\lambda + 1 - \varepsilon)} (1 - e^{-x(\lambda + 1 - \varepsilon)}) = \int_0^x dy\, e^{-y(\lambda + 1 - \varepsilon)} \int_0^\infty dq\, q e^{-\lambda q^2/2\varepsilon}$$

and remember that $\varphi$ is the Laplace transform (11), we obtain

$$f^+(x, p) = e^{-x}\varphi(\sqrt{p^2 + 2\varepsilon x}) + e^{-\varepsilon x - p^2/2} \int_0^x dy\, e^{y(\varepsilon - 1)} \int_0^\infty dq\, q\varphi(\sqrt{q^2 + 2\varepsilon y}) \tag{17}$$

In a similar fashion, we obtain for negative momenta

$$\begin{aligned} f^-(x, p) = {} & e^{-x - p^2/\varepsilon}\varphi(\sqrt{p^2 + 2\varepsilon x}) \\ & + e^{-\varepsilon x - p^2/2} \int_0^x dy\, e^{y(\varepsilon - 1)} \int_0^\infty dq\, q\varphi(\sqrt{q^2 + 2\varepsilon y}) \\ & + 2e^{-x(\varepsilon - 1) - p^2/2} \int_x^{x + p^2/2\varepsilon} dy\, e^{y(\varepsilon - 2)} \int_0^\infty dq\, q\varphi(\sqrt{q^2 + 2\varepsilon y}) \end{aligned} \tag{18}$$

The results (17) and (18) are identical to Cercignani's result for a finite spatial interval of length $L$,[7] if in his solution the incident distribution at the boundary $x = L$ is set to zero and $L$ approaches infinity.

## 4. APPLICATION TO PIASECKI'S MODEL

In this model we have $K(p, p') = \rho\, |p - p'|$. Thus, after transition to the dimensionless quantities

$$\hat{p} := \frac{p}{\sqrt{\theta m}}, \qquad \hat{x} := \rho m x, \qquad \varepsilon := -\frac{eE}{\rho\theta m} > 0$$

the eigenvalue equation (3) reads (for convenience the hats of the new variables are dropped)

$$(\varepsilon\partial_p + \lambda p - \nu(p))\, g_\lambda(p) = -M(p) \int dp'\, |p-p'|\, g_\lambda(p') \tag{19}$$

where $M(p) = (2\pi)^{-1/2} e^{-p^2/2}$ is the Maxwellian and $\nu$ is the collision frequency

$$\nu(p) := \int dp'\, |p-p'|\, M(p') = 2M(p) + |p| \left(1 - 2\int_{|p|}^{\infty} dp'\, M(p')\right) \tag{20}$$

## 4.1. Determination of the Λ-Spectrum

In order to solve (19) we closely follow Gervois' and Piasecki's approach[9] to the homogeneous equation (i.e., $\lambda = 0$). Writing

$$g_\lambda(p) =: M(p)\, G_\lambda(p) \tag{21}$$

and differentiating (19) twice with respect to $p$, we obtain

$$\varepsilon G_\lambda''' = (\nu - (\lambda - \varepsilon)\, p)\, G_\lambda'' + 2(\nu' - (\lambda - \varepsilon))\, G_\lambda' \tag{22}$$

which is a second-order differential equation for $H_\lambda := G_\lambda'$. In analogy to ref. 9, we substitute

$$G_\lambda'(p) = H_\lambda(p) =: \frac{\chi_\lambda(p)}{M(p)} \exp\left\{\frac{N(p)}{\varepsilon} - \frac{\lambda p^2}{2\varepsilon}\right\} \tag{23}$$

where $N$ (read "capital $\nu$") is the primitive function of the collision frequency

$$N(p) := \int_0^p dq\, \nu(q) = (1 + p^2) \int_0^p dq\, M(q) + pM(p) \tag{24}$$

Substitution of (23) into (22) leads to

$$\varepsilon(\chi_\lambda'' + p\chi_\lambda' - \chi_\lambda) = (\nu' - \lambda)\, \chi_\lambda - (\nu - \lambda p)\, \chi_\lambda' \tag{25}$$

From the definition (20) of the collision frequency $\nu$ we see that $\nu'' + p\nu' - \nu = 0$. Therefore $\chi_\lambda = \nu - \lambda p$ is a particular solution of (25). The second linearly independent solution can be constructed by standard methods, and we arrive at the following two solutions of (22)

$$H_\lambda^1(p) = (\nu - \lambda p) \exp\left\{\frac{N(p)}{\varepsilon} + \left(1 - \frac{\lambda}{\varepsilon}\right)\frac{p^2}{2}\right\} \tag{26a}$$

$$H_\lambda^2(p) = H_\lambda^1(p) \int_{p_0}^{p} dq \, \frac{1}{(\nu(q) - \lambda q)\, H_\lambda^1(q)} \tag{26b}$$

In order to determine the composition of the solution, let us examine the behaviour of (26a) for large $p$: insertion of (26a) into (23) and (21) leads to $g_\lambda \sim \text{const} \cdot \exp\{-(\lambda \mp 1)p^2/2\varepsilon\}$ for $p \to \pm\infty$, which grows exponentially for $p \to \infty$ if $\lambda < 1$. Moreover, for $\lambda > 1$ the solution (26b) is inacceptable because $\nu - \lambda p$ has real zeroes for $\lambda > 1$. Consequently we have

$$\lambda < 1: \quad g_\lambda(p) = M(p)\left(1 + C_\lambda \int_p^\infty dp' \, H_\lambda^1(p') \int_{p'}^\infty \frac{dq}{(\nu(q) - \lambda q)\, H_\lambda^1(q)}\right) \tag{27a}$$

$$\lambda > 1: \quad g_\lambda(p) = M(p)\left(1 + C_\lambda \int_0^p dq \, H_\lambda^1(q)\right) \tag{27b}$$

The integration constant $C_\lambda$ must be determined by insertion of (27) into the eigenvalue equation (19). If we do so for $\lambda < 1$ and evaluate at $p = 0$, we obtain

$$2C_\lambda \lambda\varepsilon \int_0^\infty dq \, \frac{M(q)\, q\nu(q)}{(\nu(q) - \lambda q)^2 (\nu(q) + \lambda q)^2} = 0$$

Since the integral is positive, this equality only holds if either $C_\lambda$ or $\lambda$ is zero. In the first case we have $g_\lambda = M$ and consequently $\lambda = \varepsilon$, in the second case $g_\lambda$ is the homogeneous stationary solution given in ref. 9.

For $\lambda > 1$, we obtain via insertion of (27b) into (19), division by $p$ and evaluation at $p = 0$:

$$\frac{\lambda - \varepsilon}{\varepsilon} = C_\lambda \left[\lambda + \int_0^\infty dq \, e^{-(\lambda - \varepsilon)q^2/2\varepsilon}\left(e^{N(q)/\varepsilon} - e^{-N(q)/\varepsilon}\right)\frac{\nu(q) - q}{2}\right] \tag{28}$$

The integrand on the right hand side of (28) is positive for $q > 0$, so that the integral is positive. Moreover, asymptotic expansion of the integrand for $q \to \infty$ and $q \to 0$ shows that the integral approaches a finite value for $\lambda \searrow 1$ and behaves like $2/\pi(\lambda - 1)$ for $\lambda \to \infty$. We conclude that $C_\lambda$ has a zero at $\lambda = \varepsilon$ and approaches $1/\varepsilon$ for $\lambda \to \infty$. A numerical evaluation of $C_\lambda$ is given in Fig. 1.

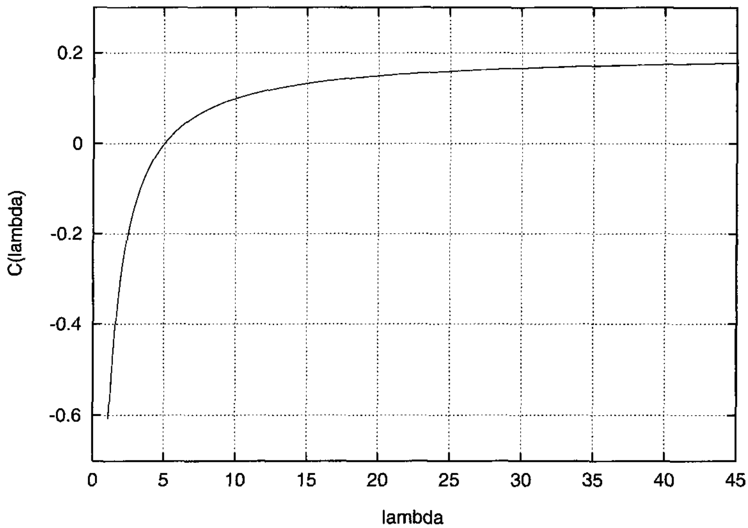


Fig. 1. The integration constant $C_\lambda$ for $\varepsilon = 5$. It has a zero at $\lambda = \varepsilon$ and approaches $1/\varepsilon = 0.2$ for $\lambda \to \infty$.

In summary, the $\lambda$-spectrum consists of the continuous interval $(1, \infty)$, with the corresponding eigenfunctions (27b) with $C_\lambda$ given by (28), and the two discrete points $\lambda = 0$, with the homogeneous solution as corresponding eigenfunction, and $\lambda = \varepsilon$ with the Maxwellian as corresponding eigenfunction. If $\varepsilon > 1$, the latter eigenvalue lies in the continuous region.

## 4.2. Half-Range Completeness

As in Section 3, we have half-range completeness in Piasecki's model as well. Once again the eigenfunction-expansion of the boundary value consists of an integral over $(1, \infty)$ plus a single contribution of the Maxwellian, which is even present for $\varepsilon > 1$.

**Theorem.** Each function $\varphi$: $\mathbb{R}^+ \to \mathbb{R}$ with $\int_0^\infty dp\, p\varphi < \infty$, that is a Laplace transform of any tempered distribution $\Phi$

$$\varphi(p) = \int_0^\infty d\mu\, \Phi(\mu)\, e^{-\mu p^2/2\varepsilon} \tag{29}$$

is a superposition of eigenfunctions

$$\varphi(p) = a_\varepsilon\, g_\varepsilon(p) + \int_1^\infty d\lambda\, A(\lambda)\, g_\lambda(p) \tag{30}$$

with expansion coefficients

$$a_\varepsilon = \sqrt{2\pi} \int_0^\infty dq\, q\varphi(q) + \sqrt{2\pi} \int_1^\infty d\mu \left( \Psi(\mu) - \frac{1}{\varepsilon} \mathscr{P} \frac{\Psi(\mu)}{C_\mu} \right) \tag{31a}$$

$$A(\lambda) = \frac{1}{\varepsilon} \sqrt{2\pi}\, \mathscr{P} \frac{\Psi(\lambda)}{C_\lambda} \tag{31b}$$

where the tempered distribution $\Psi$ is defined by

$$\int_0^\infty d\mu\, \Psi(\mu)\, e^{-\mu p^2/2\varepsilon} := e^{-N(p)/\varepsilon} \left( \varphi(p) - \int_p^\infty dq\, q\varphi(q) \right) \tag{32}$$

and the tempered distribution $T(\mu) = \mathscr{P}\Psi(\mu)/C_\mu$ is a particular solution of the (distribution-) equation $C_\mu \cdot T(\mu) = \Psi(\mu)$.

*Remarks.* (a) Because $\varphi$ is the Laplace transform (29), the distribution $\Psi$ in (32) is well defined. Moreover, from the definition of $N$ (24) we see that $e^{-N(p)/2\varepsilon} \sim e^{-p^2/2}$ for $p \to \infty$, so that we conclude from the shifting theorem of the Laplace transform(12) that $\Psi(\mu) \equiv 0$ for $\mu < 1$.

(b) According to the definition (28), $C_\lambda$ is $(\lambda - \varepsilon)$ times a strictly positive function. Hence, existence of a solution $T$ of $C_\lambda \cdot T = \Psi$ is assured by Hörmander's theorem.

*Proof.* As in Section 3.2, let us start with the assumption that $\varphi$ is a superposition of the continuous eigenfunctions alone and then prove that a contribution of the discrete eigenfunction $g_\varepsilon = M$ must be added. Thus, let us try to solve for $A(\lambda)$:

$$\varphi(p) = \int_1^\infty d\lambda\, A(\lambda)\, g_\lambda(p) = M(p) \int_1^\infty d\lambda\, A(\lambda) \left( 1 + C_\lambda \int_0^p dq\, H_\lambda^1(q) \right) \tag{33}$$

Integrating $\int_p^\infty dp'\, p'$ (33) by parts yields

$$\int_p^\infty dq\, q\varphi(q) = M(p) \int_1^\infty d\lambda\, A(\lambda) \left( 1 + C_\lambda \int_0^p dq\, H_\lambda^1(q) \right) + \int_1^\infty d\lambda\, A(\lambda) \int_p^\infty dq\, M(q)\, H_\lambda^1(q)$$

Using (33) we may replace the first term on the right hand side with $\varphi(p)$. After an evaluation of the $q$-integral in the second term we obtain

$$\psi(p) := e^{N(p)/\varepsilon} \left( \varphi(p) - \int_p^\infty dq\, q\varphi(q) \right) = \frac{\varepsilon}{\sqrt{2\pi}} \int_1^\infty d\lambda\, A(\lambda)\, C_\lambda e^{-\lambda p^2/2\varepsilon}$$

Insertion of the representation (32) for $\psi$, taking into consideration that $\Psi(\mu)=0$ for $\mu<1$ (compare remark (a)), leads to (remember uniqueness of the Laplace transform)

$$\varepsilon(2\pi)^{-1/2}C_\lambda A(\lambda)=\Psi(\lambda) \tag{34}$$

As in Section 3.2 we must distinguish two cases, depending on the electric field strength:

$\varepsilon<1$. In the case $C_\lambda$ has no zero for $\lambda>1$ and the solution of (34) is

$$A(\lambda)=\sqrt{2\pi}\,\frac{\Psi(\lambda)}{\varepsilon C_\lambda}$$

However, insertion of this result into (33) yields

$$\varphi(p)=M(p)\sqrt{2\pi}\int_1^\infty d\lambda\,\frac{\Psi(\lambda)}{\varepsilon C_\lambda}+\varphi(p)$$
$$-M(p)\sqrt{2\pi}\left(\int_1^\infty d\lambda\;\Psi(\lambda)+\int_0^\infty dq\;q\varphi(q)\right)$$

We see that in (33) the term $a_\varepsilon M(p)$, where $a_\varepsilon$ is given by (31a), must be added. This yields the theorem.

$\varepsilon>1$. In this case $C_\lambda$ has a zero at $\lambda=\varepsilon$ and the general solution of (34) in the space of tempered distributions is

$$A(\lambda)=\sqrt{2\pi}\,\mathscr{P}\,\frac{\Psi(\lambda)}{\varepsilon C_\lambda}+c_\varepsilon\delta(\lambda-\varepsilon)$$

where the first term is a particular solution of (34) and $c_\varepsilon$ is an arbitrary constant. Insertion of this result into (33) yields

$$\varphi(p)=M(p)\sqrt{2\pi}\int_1^\infty d\lambda\,\mathscr{P}\,\frac{\Psi(\lambda)}{\varepsilon C_\lambda}+\varphi(p)$$
$$-M(p)\sqrt{2\pi}\left(\int_1^\infty d\lambda\;\Psi(\lambda)+\int_0^\infty dq\;q\varphi(q)\right)+c_\varepsilon M(p)$$

If we identify $c_\varepsilon$ with $a_\varepsilon$ in (31a), this equality holds and the theorem is proven. ∎

### 4.3. Explicit Form of the Solution

For any boundary value $\varphi$ satisfying the conditions of the half-range completeness theorem, the solution of the stationary half-space problem is (see (4))

$$f(x, p) = a_\varepsilon M(p)\, e^{-\varepsilon x} + \int_1^\infty d\lambda\, A(\lambda)\, e^{-\lambda x} g_\lambda(p) \tag{35}$$

Again we can extend the solution to boundary values $\varphi$ beyond the assumptions of the half-range completeness theorem by explicit evaluation of the integrals in (35). Straightforward insertion of the eigenfunctions (27b) and expansion coefficients (31) yields

$$\begin{aligned} f(x, p) = \sqrt{2\pi}\, M(p)\, e^{-\varepsilon x} \bigg\{ &\int_0^\infty dq\, q\varphi(q) \\ &+ \int_1^\infty d\lambda\, \Psi(\lambda) + \int_1^\infty d\lambda\, \mathscr{P} \frac{\Psi(\lambda)}{\varepsilon C_\lambda} (e^{-(\lambda-\varepsilon)x} - 1) \\ &+ \int_1^\infty d\lambda\, e^{-(\lambda-\varepsilon)x}\, \Psi(\lambda) \int_0^p dq\, (\partial_q e^{N(q)/\varepsilon - \lambda q^2/2\varepsilon})\, e^{q^2/2} \bigg\} \end{aligned}$$

In the third term in the curly braces we write

$$e^{-(\lambda-\varepsilon)x} - 1 = -(\lambda - \varepsilon) \int_0^x dy\, e^{-(\lambda-\varepsilon)y}$$

Then we obtain after insertion of $C_\lambda$ accoding to (28) and using (32)

$$\begin{aligned} f(x, p) = e^{-\varepsilon x - p^2/2} \bigg\{ &\int_0^\infty dq\, q\varphi(q) - \varepsilon \int_0^x dy\, e^{\varepsilon y} \psi(\sqrt{2\varepsilon y}) \\ &- \int_0^x dy\, e^{\varepsilon y} \int_0^\infty dq\, e^{q^2/2} (e^{N(q)/\varepsilon} - e^{-N(q)/\varepsilon}) \frac{v(q) - q}{2} \psi(\sqrt{2\varepsilon y + q^2}) \bigg\} \\ &+ e^{N(p)/\varepsilon} \psi(\sqrt{2\varepsilon x + p^2}) - e^{-p^2/2} \int_0^p dq\, q e^{N(q)/\varepsilon + q^2/2} \psi(\sqrt{2\varepsilon x + q^2}) \end{aligned} \tag{36}$$

where the function $\psi$ is defined (see (32))

$$\psi(p) := e^{-N(p)/\varepsilon} \left( \varphi(p) - \int_p^\infty dq\, q\varphi(q) \right)$$

## 5. SUMMARY AND CONCLUSIONS

In this article we have seen that Case's eigenfunction technique may be applied to half-space problems of kinetic equations even in the presence of a constant external field. For two particular models, half-range completeness of the eigenfunctions has been proven for a large class of boundary values.

In both models, the solution of the half-space problem is a superposition of the eigenfunctions belonging to the continuous part of the $\lambda$-spectrum plus a single contribution of the Maxwellian, which belongs to the eigenvalue $\lambda = \varepsilon$. The latter contribution is present even if $\varepsilon$ lies in the range of the continuous spectrum. This confirms a conjecture of Stichel and Strothman[13]: the assumption that $\lambda = \varepsilon$ is a singular eigenvalue was their starting point for an asymptotic analysis of the boundary value problem of the Boltzmann equation. The question arises whether this is a general feature of the spectrum.

A more fundamental open question is which general class of collision models is amenable to the eigenfunction technique in the presence of an electric field. An obviously necessary condition is that the collision frequency $\nu := \int KM$ must not increase more than linearly in $p$ as $p \to \infty$, for otherwise the eigenvalue equation has no $L_1$-integrable solution. It would be desirable to establish sufficient conditions under which half-range completeness holds. For general collision kernels however, we can not hope that the eigenfunction-solution can be extended to boundary values $\varphi$ not satisfying the assumptions in the half-range completeness theorem, even though this was shown possible in the collision models considered in this article (see Sections 3.3 and 4.3). This may be possible only for particular choices of the collision kernel.

## ACKNOWLEDGMENT

The author is grateful to the "Studienstiftung des deutschen Volkes" for financial support of this work.

## REFERENCES

1. C. Cercignani, *The Boltzmann Equation and Its Application*, Springer, New York, 1986.
2. K. M. Case, Elementary solutions of the transport equation and their applications, *Annals of Physics* **9**:1 (1960).
3. P. F. Zweifel, A generalized transport equation, *Transport Theory Stat. Phys.* **11**:183 (1983).
4. W. Greenberg and C. V. M. van der Mee, Abstract kinetic equations relevant to supercritical media, *J. Func. Anal.* **57**:111 (1984).

5. C. Dalitz, Exact solutions of the semiconductor Boltzmann equation, *Physica A* **203**:125 (1994).
6. C. Dalitz, Initial-boundary value problems of the Boltzmann equation at zero temperature. *Transport Theory Stat. Phys.* (1997), in press.
7. C. Cercignani, An explicitly solvable kinetic model for semiconductors, *J. Stat. Phys.* **77**:1039 (1994).
8. J. Piasecki, Approach to field-induced stationary state in a gas of hard rods, *J. Stat. Phys.* **30**:185 (1983).
9. A. Gervois and J. Piasecki, Stationary velocity distribution in an external field: a one-dimensional model, *J. Stat. Phys.* **42**:1091 (1986).
10. O. J. Eder and M. Posch, Solution of the one-dimensional linear Boltzmann equation for charged Maxwellian particles in an external field, *J. Stat. Phys.* **52**:1031 (1988).
11. L. Hörmander, *The analysis of linear partial differential operators, Part 1*, Springer, Berlin (1983).
12. G. Doetsch, *Handbuch der Laplacetransformation, Teil 1*, Birkhäuser, Basel (1970).
13. P. C. Stichel and D. Strothmann, Asymptotic analysis of the high field semiconductor equation, *Physica A* **202**:553 (1994).